

BOOLEAN DISTANCE FOR GRAPHS

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The boolean distance between two points x and y of a connected graph G is defined as the set of all points on all paths joining x and y in G (\emptyset if $x = y$). It is determined in terms of the block-cutpoint graph of G , and shown to satisfy the triangle inequality $b(x, y) \subseteq b(x, z) \cup b(z, y)$. We denote by $B(G)$ the collection of distinct boolean distances of G and by $M(G)$ the multiset of the distances together with the number of occurrences of each of them. Then $|B(G)| = 1 + \binom{b}{2}$ where b is the number of blocks of G . A combinatorial characterization is given for $B(T)$ where T is a tree. Finally, G is reconstructible from $M(G)$ if and only if every block of G is a line or a triangle.

1. Boolean distance

All notation and terminology in this paper not defined below can be found in [1]. In particular a path does not have repeated points. If G is a connected graph, we define the *boolean distance* $b(x, y)$ between points x and y of G as follows: if $x = y$, then $b(x, y) = \emptyset$, and if $x \neq y$, then $b(x, y)$ is the set of all points on all paths joining x and y . The boolean distances of G can be determined by its block structure, as will be shown below. To this end recall that the block-cutpoint graph of G , $bc(G)$, is the bipartite graph having as points the blocks and the cutpoints of G , in which block b is adjacent to cutpoint c if and only if $c \in b$ in G . For any point x of G , let $b(x)$ be x itself if x is a cutpoint of G and the unique block of G containing x if not. Since $bc(G)$ is a tree [1, p. 37], for any points x, y of G there is a unique path joining $b(x)$ and $b(y)$ in $bc(G)$, which will be denoted by $P(x, y)$. The study of the cutpoints of G on $P(x, y)$ suggested the concept of a “cutting

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center" of a tree in [2]. The following result relating the boolean distances of G to its block structure can now be stated.

Theorem 1. *For any distinct points x, y of G , $b(x, y)$ is the union of all blocks of G (considered as point-sets) lying on $P(x, y)$ in $bc(G)$.*

Proof. The path $P(x, y)$ has the form $c_0, b_1, c_1, b_2, \dots, c_{n-1}, b_n, c_{n+1}$ where the c_i are cutpoints and the b_i are blocks of G such that $c_i \in b_{i-1} \cap b_i$. The first cutpoint c_0 appears only if x is a cutpoint and then $c_0 = x$, otherwise $x \in b_1$, and similarly at the other end. First we prove the inclusion $b(x, y) \subseteq b_1 \cup \dots \cup b_n$. If a path of G leaves a block, it cannot return to this block, because that would necessitate repeating a cutpoint. Therefore if P is any path joining x and y in G , then the sequence of blocks and cutpoints encountered by P is a path joining $b(x)$ and $b(y)$ in $bc(G)$. But the latter path must be $P(x, y)$, and so all the points of P are contained in $b_1 \cup \dots \cup b_n$. Now we prove the opposite inclusion $b_1 \cup \dots \cup b_n \subseteq b(x, y)$. Let z be any point of b_i . Then by [1, p. 28] G has a path P joining c_i and c_{i+1} and containing z (if $i = 1$ and c_0 does not appear, then G has a path P joining x and c_1 and containing z , and similarly at the other end). Let Q be any path joining x and c_i and R any path joining c_{i+1} and y in G . Then by the previous argument, Q followed by P followed by R is a path in G , and this path joins x and y and contains z . \square

As a corollary we can see that $b(x, y)$ is a *boolean metric* in the sense of [4].

Corollary 1a. (1) $b(x, y) = \emptyset$ if and only if $x = y$.

(2) $b(x, y) = b(y, x)$.

(3) $b(x, y) \subseteq b(x, z) \cup b(z, y)$.

Proof. The first two statements are obvious, and third follows from Theorem 1. In fact for $x \neq y$ there is equality in (3) if and only if $b(z)$ appears in $P(x, y)$. \square

2. Distance sets

The set of all boolean distances between points of G is called the *distance set* of G and is denoted by $B(G)$; it is understood that \emptyset is always included as a boolean distance. Obviously $|B(G)| = 2$ if and only if G is a block. If G contains a cycle, boolean distances between distinct point-pairs may be equal. We write p for the number of points of G and b for the number of blocks, trusting that there will be no confusion between the symbols b and $b(x, y)$.

Theorem 2. *If G is a connected graph with b blocks, then $|B(G)| = 1 + \binom{b+1}{2}$. In particular $|B(G)| = 1 + \binom{p}{2}$ if and only if G is a tree.*

Proof. By Theorem 1, $B(G) - \{\emptyset\}$ is the set of unions of blocks of G (considered as point-sets) lying on paths of $bc(G)$ beginning and ending in blocks of G .

Therefore $|B(G)| - 1$ is equal to b (single blocks) plus $\binom{b}{2}$ (paths joining distinct blocks). The result on trees follows from this and from the fact that a connected graph has $p - 1$ blocks if and only if it is a tree (certainly a tree has $p - 1$ blocks, and if new lines are added to a tree, the number of blocks first decreases and then never increases). \square

We remark that for almost all graphs G on p points $|B(G)| = 2$ as $p \rightarrow \infty$, as it is observed in [3, p. 207] that almost all graphs are blocks. We also note that when $p \geq 3$, G is a star if and only if $B(G) - \{\emptyset\}$ contains only sets with two or three points. The next theorem characterizes the distance sets of trees.

Theorem 3. *Let X be an n -element set and let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a collection of $\binom{n}{2}$ subsets of X . Then there exists a tree T with point-set X and $B(T) - \{\emptyset\} = \mathcal{F}$ if and only if the following three conditions are fulfilled:*

- (i) *For any $F \in \mathcal{F}$, $|F| \geq 2$.*
- (ii) *Any set F in \mathcal{F} contains exactly $|F| - 1$ 2-element subsets of \mathcal{F} . These 2-element subsets have the form $\{x_1, x_2\}$, $\{x_2, x_3\}$, \dots , $\{x_{k-1}, x_k\}$, where $\{x_1, x_2, \dots, x_k\} = F$. We call x_1 and x_k end-elements of F .*
- (iii) *If $F_1, F_2 \in \mathcal{F}$ and $F_1 \cap F_2 = \{x\}$ where x is an end-element of both F_1 and F_2 , then $F_1 \cup F_2 \in \mathcal{F}$.*

Proof. The necessity is obvious. In order to prove the sufficiency of the conditions, construct a graph G having point-set equal to X and line-set equal to the family of 2-element subsets of \mathcal{F} . Then G has no cycles, for if x_0, x_1, \dots, x_{r-1} were the points of a cycle of G in that order, then $\{x_i, x_{i+1}\} \in \mathcal{F}$ for each i (indices mod r), hence by repeated use of (iii), $\{x_0, x_1, \dots, x_{r-1}\} \in \mathcal{F}$. Then by (ii) $\{x_0, x_1, \dots, x_{r-1}\}$ would have to contain exactly $r - 1$ lines of G , but it contains at least r of them, a contradiction showing that G has no cycles. Now if any two points of G appeared more than once as end-elements, then by a standard argument G would contain a cycle, which is impossible. Hence there appear at most $\binom{n}{2}$ pairs of end-elements, so $|\mathcal{F}| \leq \binom{n}{2}$. But by assumption $|\mathcal{F}| = \binom{n}{2}$, and it follows that every two points of G appear as end-elements, and G is connected. Thus G is a tree and the point-sets of its paths are precisely the singletons and the members of \mathcal{F} . Hence $B(G) - \{\emptyset\} = \mathcal{F}$. \square

3. Reconstructibility from boolean distances

The collection of boolean distances of G can be regarded as a multiset by taking the multiplicity of the sets of points into account. For example, \emptyset has multiplicity p and the set of endpoints of a bridge has multiplicity 1. We thus define the *boolean distance multiset* $M(G)$ as the pair $(B(G), m)$, where m is the

function

$$m : B(G) \rightarrow \left\{ 1, 2, \dots, \binom{p}{2} \right\}.$$

that associates with each set $S \in B(G)$ the number of unordered pairs $\{x, y\}$ of points of G such that $b(x, y) = S$. A graph G with given point-set is said to be *reconstructible from its boolean distance multiset* if G is uniquely determined by $M(G)$, i.e., there is a procedure to identify the lines of G using only $M(G)$.

We now determine the multiplicities of the blocks of G considered as point-sets.

Theorem 4. *A set $S \in B(G)$ has multiplicity $m(S) = \binom{|S|}{2}$ if and only if S induces a block of G .*

Proof. Clearly we may assume $S \neq \emptyset$. Then by Theorem 1, $bc(G)$ has a unique path of the form $b_1, c_1, \dots, c_{n-1}, b_n$, where the b_i are blocks and the c_i cutpoints of G , such that $S = b_1 \cup \dots \cup b_n$. Thus S induces a block of G if and only if $n = 1$. If $n = 1$, then

$$m(S) = \binom{|b_1|}{2} = \binom{|S|}{2}.$$

If $n = 2$, then

$$m(S) = \frac{1}{2}(|b_1| - 1)(|b_2| - 1) < \binom{|S|}{2}.$$

If $n \geq 3$, then

$$m(S) = \frac{1}{2}|b_1| \cdot |b_n| < \binom{|S|}{2}. \quad \square$$

We define the *block completion* $K(G)$ as the graph obtained by replacing each block of G by a complete subgraph on the same set of points. Thus $K(G)$ is a 'block graph': see [1, p. 29]. Obviously G and $K(G)$ have the same cutpoints. We then have the following corollary of Theorem 4.

Corollary 4a. *For any connected graph G , the block completion $K(G)$ is reconstructible from the multiset $M(G)$.*

Proof. The blocks of G are uniquely determined from the condition $m(S) = \binom{|S|}{2}$, and then two points are adjacent in $K(G)$ if and only if they belong to the same block of G . \square

We conclude with the following corollary showing which graphs G are reconstructible from $M(G)$.

Corollary 4b. *A connected graph G is reconstructible from $M(G)$ if and only if G has no cycle of length greater than 3.*

Proof. Assume that G contains a cycle C_n of length $n \geq 4$. Then C_n is contained in some block H having at least four points. If H is complete we denote by G_1 the graph obtained from G by deleting an arbitrary line of H . If H is not complete we denote by G_1 the graph obtained from G by adding a line between two nonadjacent points of H . In both cases G and G_1 have the same cutpoints and blocks (considered as point-sets). Hence $bc(G) = bc(G_1)$ and $M(G) = M(G_1)$, so G is not reconstructible from $M(G)$. Conversely, assume that G has no cycle of length greater than 3. We show that all blocks of G are lines or triangles. For otherwise there is a block H with at least four points and the longest cycle of H contains exactly three points, say x, y and z . Then z , say, is adjacent to a fourth point t of H , and there is a path $P = (t, \dots, y)$ not containing z . If x is not a point of P , then (t, \dots, y, x, z, t) is a cycle of length greater than 3, and if x is a point of P , then (t, \dots, x, y, z, t) is such a cycle. This contradiction proves that the blocks of G are lines or triangles. Therefore $K(G) = G$ and by Corollary 4a, G is reconstructible from $M(G)$. \square

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